Introduction to circulating atmospheres

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The governing physical laws

The aim of this chapter is to introduce the basic physical laws which govern the circulation of the atmosphere and to express them in convenient mathematical forms. No attempt is made at either completeness or rigour beyond the requirements of the later chapters. Those who wish for a more detailed discussion are referred to one of the many excellent texts on dynamical meteorology which are now available. Those by Holton (1992) and by Gill (1982) are particularly recommended.

1.1 The first law of thermodynamics

The first law may be stated simply in its qualitative form: heat is a form of energy. The transformation of heat energy into various forms of mechanical energy is the process which drives the global circulation of the atmosphere and which is responsible for the formation of the weather systems whose cumulative effects define the climate of a particular region. These transformations will be discussed in more detail in Chapter 3. In this section, the first law will be expressed in mathematical terms. But, first, it will be necessary to consider the thermodynamic properties of the air which makes up the atmosphere.

The 'thermodynamic state' of a parcel of air is defined by specifying its composition, pressure, density, temperature, and so on. In fact, these properties are not independent of one another, but are related through the 'equation of state' of the air.

For our purposes, only one constituent of the air varies significantly, and that is water vapour. The remaining gases which make up the bulk of the atmosphere are present in constant proportions, at least up to very great heights. These are principally nitrogen and oxygen, with smaller concentrations of argon and carbon dioxide. Other gases are present in very

Table 1.1. *Composition of dry air*

Gas	Volume mixing ratio
Nitrogen (N_2)	0.780 83
Oxygen (O_2)	0.209 47
Argon (Ar)	0.009 34
Carbon dioxide (CO_2)	0.000 33

small amounts; some are important in determining the transparency of the atmosphere to various frequencies of electromagnetic radiation, and some play a crucial role in the chemistry of the atmosphere. But for our purposes, we may ignore them. Table 1.1 summarizes the normal composition of dry air.

We will return to water vapour shortly. If we consider 'dry air', then its pressure p, temperature T and density ρ are related by the 'ideal gas law':

$$p = \rho R T. \tag{1.1}$$

This equation of state needs modification at very high pressures and low temperatures. But over the range of temperature and pressure encountered in the atmosphere, it is perfectly adequate. The gas constant R is related to the universal gas constant R^* by

$$R = R^*/m. \tag{1.2}$$

where m is the mean (by volume) molecular weight of the cocktail of gases comprising dry air. The equation of state, Eq. (1.1), means that it is only necessary to know any two of p, T or ρ to specify the thermodynamic state of the air completely. It is sometimes more convenient to work with 'specific volume' $\alpha = 1/\rho$ (i.e. the volume occupied by a unit mass of air) rather than with ρ.

The temperature of the air is simply a measure of the 'internal energy' of the air, that is, of the energy which is associated with the random motion of the molecules and possibly with their rotation and internal vibration. If two masses of gas are brought into intimate contact, this internal energy is rapidly shared between them and their temperatures become equal. When their temperatures are unequal, a flow of heat from the hotter to the colder mass takes place. An infinitesimal change of the internal energy U of a unit mass of dry air is related to the temperature change by:

$$\mathrm{d}U = c_v \mathrm{d}T, \tag{1.3}$$

where c_v is the 'specific heat at constant volume'.

If an infinitesimal quantity of heat $\mathrm{d}Q$ is added to an element of air, it may contribute to an increase in its internal energy, or it may be converted into mechanical energy, or a combination of the two. But the change of internal energy plus the mechanical work done must balance the heat added. This is the mathematical statement of the first law of thermodynamics:

$$\mathrm{d}Q = \mathrm{d}U + \mathrm{d}W. \tag{1.4}$$

Typically, work is done by the air parcel when it expands against the pressure exerted by the surrounding gas. Assuming that the pressure of the element of gas is equal to the pressure of its surroundings (always true if the expansion is gentle), the work done is related to the change of volume:

$$\mathrm{d}W = p\mathrm{d}\alpha. \tag{1.5}$$

Thus a working form of the first law of thermodynamics can be written:

$$\mathrm{d}W = c_v\mathrm{d}T + p\mathrm{d}\alpha. \tag{1.6}$$

A more convenient form is obtained using the equation of state, Eq. (1.1):

$$\mathrm{d}Q = c_p\mathrm{d}T - \alpha\mathrm{d}p, \tag{1.7}$$

where $c_p = c_v + R$ is the 'specific heat at constant pressure'. This form is useful since many atmospheric processes occur more nearly at constant pressure than at constant volume.

A 'thermodynamic process' is a slow change of the thermodynamic state of an element of air; it may be described by a curve on a 'thermodynamic diagram' on which any two of the state variables are plotted. A particularly important class of thermodynamic processes is the 'adiabatic' process, in which no heat enters or leaves the element. From Eq. (1.7),

$$c_p\mathrm{d}T = \alpha\mathrm{d}p, \tag{1.8}$$

during an adiabatic process, or, integrating,

$$T = \theta(p/p_R)^\kappa,\ \kappa = R/c_p, \tag{1.9}$$

where θ is a constant of integration which may be interpreted simply as the temperature at pressure p_R during the adiabatic process; θ is generally called the 'potential temperature' and p_R is usually (but arbitrarily) taken to be 100 kPa. Indeed, the potential temperature may be regarded as a new thermodynamic variable and Eq. (1.9) as an alternative equation of state.

Yet another form of the first law is obtained if Eq. (1.7) is written in terms of potential temperature:

$$\mathrm{d}Q = c_p T\,\mathrm{d}(\ln\theta). \tag{1.10}$$

Finally, if the heat is added over a time dt, the rate of change of potential temperature of the element is:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{c_p}(p/p_R)^{-\kappa}\frac{\mathrm{d}Q}{\mathrm{d}t} = \mathscr{Q}. \tag{1.11}$$

The term dQ/dt is sometimes called the 'diabatic warming rate'; $\mathscr{Q}$ denotes the rate of change of θ due to heating. This is the rate of change of θ when a particular fluid element is followed, and is more usually written $\mathrm{D}\theta/\mathrm{D}t$, the 'Lagrangian derivative'. This differs from the 'Eulerian derivative', which measures the rate of change at a fixed point in space. If the gradient of θ at any instant is $\nabla\theta$, then the difference between the Eulerian and Lagrangian derivatives is simply the rate of change due to advection, $-\mathbf{u}\cdot\nabla\theta$. Thus:

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \mathscr{Q} \tag{1.12}$$

The quantity of moisture in the air may be measured by the mass mixing ratio of water vapour $r = \rho_v/\rho_d$, ρ_v being the mass of water vapour in a unit volume and ρ_d the mass of dry air in the same volume. The saturation mixing ratio r_s is a function of temperature and pressure of the air, and may be as large as 0.030 in the warmest parts of the tropics. Generally, it is much less, with a typical value of r_s of 0.010 at the surface. For an average atmospheric temperature of 255 K and pressure of 50 kPa, $r_s = 0.005$. The equation of state of moist air is obtained by writing the total pressure as the sum of the vapour pressure and the partial pressure of the dry air, the ideal gas equation applying to both components separately with a suitable gas constant. The result can be written:

$$p = R_d\frac{(1 + (R_v/R_d)r)}{(1 + r)}\rho T. \tag{1.13}$$

In fact, for most of the atmosphere, the difference between the equation of state for moist air and that for dry air is not very large, and may frequently be ignored when discussing the large scale circulation. The primary importance of the variable moisture content of air is the huge latent heat of condensation of water vapour, larger than that of any other common substance, which means that very large amounts of heat are released when water condenses. Equally, large amounts of heat must be supplied when water evaporates. A quantity of heat

$$dQ = -L dr \tag{1.14}$$

is released when the mixing ratio is reduced by condensation, where L is the latent heat of condensation. Thus if 10 mm of rain falls during a 24 hour period, the release of latent heat amounts to 289 W m^2, which is comparable to the typical insolation per unit area.

An equation describing the evolution of the humidity mixing ratio is analogous to the equation of conservation of energy. It is simply based on the hypothesis that any change of the moisture content of an air parcel is due to a rate of evaporation E into the parcel, or of condensation P taking water vapour out of the parcel. Small amounts of water are created or destroyed by chemical reactions, but these can generally be neglected. For our purposes, it is often enough to suppose that any condensed water falls out of the air immediately as rain, though some sophisticated models carry the suspended liquid and solid water content of the air as separate variables. Then

$$\frac{\partial r}{\partial t} + \mathbf{u} \cdot \nabla r = E - P. \tag{1.15}$$

The Lagrangian rate of change of water mixing ratio leads to an important contribution to the heating rate:

$$S = -L\frac{Dr}{Dt} = L(P - E). \tag{1.16}$$

This term is frequently dominant in the Earth's atmosphere, particularly in localized regions of persistent rainfall.

1.2 Conservation of matter

Consider some fixed volume of space V, enclosed by a surface A. The mass of air enclosed in this volume is:

$$m = \int_V \rho d\tau. \tag{1.17}$$

Any change in this mass must be accomplished by a flux of mass into or out of the volume, so that

$$\frac{\partial}{\partial t}\int_V \rho d\tau = -\int_A \rho\mathbf{u} \cdot \mathbf{n} dA = -\int_V \nabla \cdot \rho\mathbf{u} d\tau, \tag{1.18}$$

where the divergence theorem has been used. Since this must apply to any arbitrary volume, the two integrands in the volume integrals must be equal, so that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{1.19}$$

This is the full form of the 'equation of continuity'. It may be simplified further if the density is broken into a reference profile ρ_R, which represents the mean density at any height and depends only on height, and the departure ρ_A from this reference density. For flow in planetary atmospheres, the variation of density in the vertical is very much larger than any horizontal fluctuations. Then scale analysis shows that:

$$\left| \frac{\partial \rho}{\partial t} \right| \ll |\nabla \cdot (\rho \mathbf{u})| , \tag{1.20}$$

so that the continuity equation can be reduced to:

$$\nabla \cdot \mathbf{v} + \frac{1}{\rho_R} \frac{\partial \rho_R w}{\partial z} = 0. \tag{1.21}$$

This result would become invalid if the flow speed approached the sound speed, in which case the full continuity equation, Eq. (1.19), must be used.

1.3 Newton's second law of motion

Newton's second law of motion is used to calculate how the motion of the atmosphere evolves. It states that the acceleration of a parcel of air of unit mass is equal to the vector sum of the forces acting upon it, that is,

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \sum_i \mathbf{F}_i, \tag{1.22}$$

This is frequently called the 'equation of motion' or the 'momentum equation'. The forces which we need to consider in the case of atmospheric motion are:

(i) The gravitational force. We consider this to be a constant vector $\mathbf{g}$ directed towards the centre of the Earth. It can be written as the gradient of a 'gravitational potential' $\nabla\Phi$.

(ii) The pressure gradient force. Figure 1.1 shows two surfaces of constant pressure a distance Δs apart. Consider a small volume of air, cross sectional area ΔA between them. The mass of air in the volume is $\rho \Delta A \Delta s$ and the net force due to the pressure of the surrounding air is

$$p_0 \Delta A - \left(p_0 + \left| \frac{\partial p}{\partial s} \right| \Delta s \right) \delta A = - \left| \frac{\partial p}{\partial s} \right| \Delta s \Delta A. \tag{1.23}$$

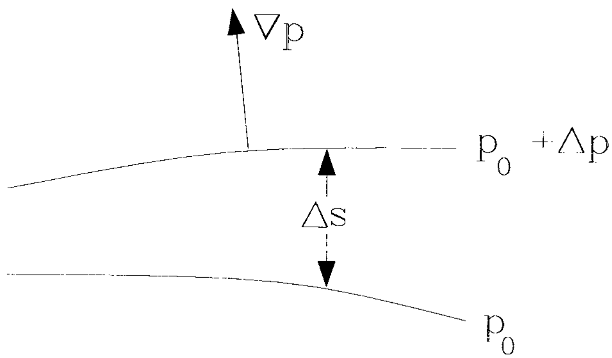

Fig. 1.1. The pressure gradient force.

The pressure gradient force per unit mass is therefore:

$$\mathbf{F}_p = -\frac{1}{\rho}\nabla p. \tag{1.24}$$

(iii) The friction force. Friction is generally a result of turbulent exchanges of momentum between the Earth's surface and the overlying layers of air. Accurate simple formulae for this transfer do not exist, and rather complex empirical relationships have to be employed in global circulation models. Generally, we will simply call the friction force $\mathscr{F}$, and note that it will usually act in such a direction as to reduce the wind towards rest. A very approximate linear parametrization of friction will be used on occasions where an analytical expression for friction is needed:

$$\mathscr{F} = -\frac{\mathbf{v}}{\tau_D}, \tag{1.25}$$

where τ_D is a drag or 'spin up' timescale. Such a term represents an exponential decay of the velocity towards zero in the absence of other forces. It is sometimes called 'Rayleigh friction'. A typical global mean spin up time for the Earth's atmosphere is around five days.

Equation (1.22) describes the acceleration of a parcel of air in an inertial frame of reference, that is, in a frame of reference which is not accelerating and which is therefore not rotating. It is usual to describe motion in the atmosphere relative to a noninertial frame of reference which is embedded in the rotating Earth. The relationship between acceleration in an inertial frame of reference, denoted I, and in a uniformly rotating frame of reference, denoted R, is derived, for example, in Pedlosky, page 17; the result is

$$\left(\frac{\mathrm{D}\mathbf{u}_I}{\mathrm{D}t}\right)_I = \left(\frac{\mathrm{D}\mathbf{u}_R}{\mathrm{D}t}\right)_R - \nabla\left(\frac{|\mathbf{\Omega}\times\mathbf{r}|^2}{2}\right) + 2\mathbf{\Omega}\times\mathbf{u}_R. \tag{1.26}$$

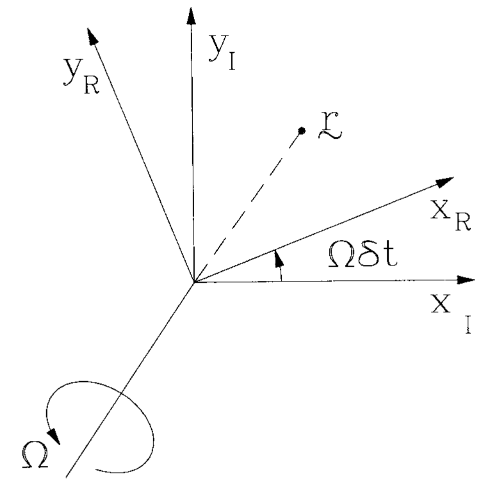

Fig. 1.2. A uniformly rotating frame of reference

Figure 1.2 illustrates the notation. $\mathbf{u}_I$ is the velocity in an inertial frame and $\mathbf{u}_R$ is the velocity in a rotating frame. From now on, the velocities and derivatives without any such subscript will be assumed to refer to a frame which is rotating with the solid Earth.

The second term on the right hand side of Eq. (1.26) is the centripetal acceleration. Since it is the gradient of a scalar, it introduces no structural change to the equation of motion; it can be absorbed into the definition of gravitational potential. The centripetal acceleration makes a very small correction to the gravitational acceleration, which is largest at the equator.

Thus, Newton's second law may be written:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = 2\boldsymbol{\Omega} \times \mathbf{u} - \frac{1}{\rho}\nabla p + \mathscr{F}. \tag{1.27}$$

This has now been written in terms of the Eulerian rate of change of velocity. The first term on the left hand side arises from the rotation of the frame of reference and is a most important term for global scale circulations. It is sometimes called the 'Coriolis force'. Strictly, it should be regarded as a 'pseudo-force', that is, a mental construct which is designed to make it appear that Newton's second law is holding despite the rotation of the frame of reference. Note that since the Coriolis force always acts at right angles to the fluid motion, it can do no work. Acting in isolation from other forces, it will cause parcel trajectories to be circular, with radius $|\mathbf{u}|/(2|\boldsymbol{\Omega}|)$. Such motion is termed 'inertial flow'.

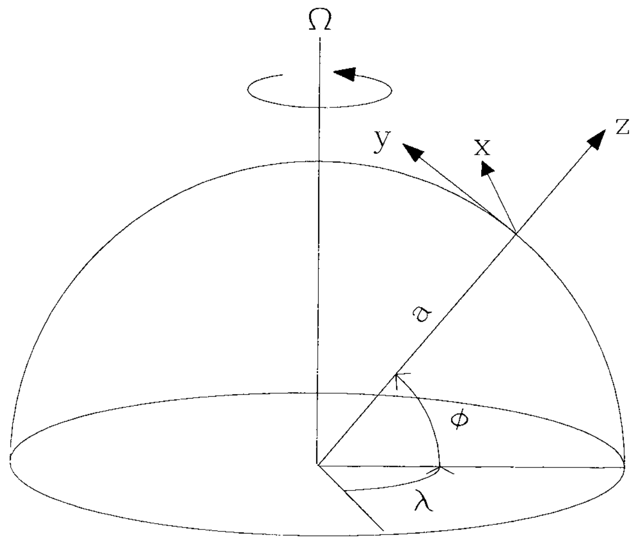

Fig. 1.3. Coordinates used to describe atmospheric motions relative to a spherical planet.

1.4 Coordinate systems

Up to now, the equations governing the atmospheric circulation have been expressed in a general vector notation. But for practical purposes, they must be written in terms of the components of velocity, etc., in orthogonal directions. This will lead us to consider the very marked asymmetry between the vertical and horizontal directions, and thereby to simplify the equations, obtaining a usable set for computation.

The Earth is nearly a sphere, and so it is natural to employ spherical coordinates ϕ (latitude), λ (longitude) and r (distance from the centre of the Earth). In fact, it is possible (though not trivial) to show that the slightly oblate shape of the Earth can be ignored, and that its effect can be represented by small variations of $\mathbf{g}$, the acceleration due to gravity, with latitude if necessary. Recognizing that the depth of the atmosphere is very small compared to the radius of the Earth a, we write:

$$r = a + z, \text{ with } z \ll a, \tag{1.28}$$

where z is the height above mean sea level. The three components of velocity are denoted u (zonal), v (meridional) and w (vertical). Figure 1.3 defines the notation. The equations of motion are derived in general curvilinear coordinates in standard texts on fluid dynamics such as Batchelor (1967). The results are quoted here for reference:

Equations of motion:

$$\frac{\partial u}{\partial t}+\frac{u}{a\cos\phi}\frac{\partial u}{\partial\lambda}+\frac{v}{a}\frac{\partial u}{\partial\phi}+w\frac{\partial u}{\partial z}+\frac{uv}{a}\tan\phi+\frac{uw}{a}$$
$$=2\Omega v\sin\phi-2\Omega w\cos\phi-\frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda}+\mathscr{F}_1, \tag{1.29a}$$

$$\frac{\partial v}{\partial t}+\frac{u}{a\cos\phi}\frac{\partial v}{\partial\lambda}+\frac{v}{a}\frac{\partial v}{\partial\phi}+w\frac{\partial v}{\partial z}+\frac{u^2}{a}\tan\phi+\frac{vw}{a}$$
$$=-2\Omega u\sin\phi-\frac{1}{\rho a}\frac{\partial p}{\partial\phi}+\mathscr{F}_2, \tag{1.29b}$$

$$\frac{\partial w}{\partial t}+\frac{u}{a\cos\phi}\frac{\partial w}{\partial\lambda}+\frac{v}{a}\frac{\partial w}{\partial\phi}+w\frac{\partial w}{\partial z}-\frac{(u^2+v^2)}{a}$$
$$=2\Omega u\cos\phi-g-\frac{1}{\rho}\frac{\partial p}{\partial z}+\mathscr{F}_3. \tag{1.29c}$$

Equation of continuity:

$$\frac{1}{a\cos\phi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\phi}\frac{\partial(v\cos\phi)}{\partial\phi}+\frac{1}{\rho_R}\frac{\partial(\rho_R w)}{\partial z}=0. \tag{1.30}$$

Thermodynamic equation:

$$\frac{\partial\theta}{\partial t}+\frac{u}{a\cos\phi}\frac{\partial\theta}{\partial\lambda}+\frac{v}{a}\frac{\partial\theta}{\partial\phi}+w\frac{\partial\theta}{\partial z}=\mathscr{Q}. \tag{1.31}$$

If the meridional extent of the motion is limited, it is often advantageous to use a local Cartesian set of coordinates (x, y, z), where $y = a(\phi - \phi_0)$ is the distance poleward from some reference latitude and $x = a\lambda\cos\phi$ is the eastward distance along the latitude circle. Such a coordinate system, neglecting many of the curvature terms in Eqs. (1.29a) – (1.31), simplifies the equations without removing any of the primary physical processes they represent. While it is inadequate for exact work, such as constructing numerical models of the global circulation or constructing budgets of global or zonal mean quantities, it often very helpful for expository purposes and will be used frequently in later chapters.

1.5 Hydrostatic balance and its implications

The vertical component of the momentum equation is dominated by the vertical pressure gradient term and the acceleration due to gravity. These are many orders of magnitude larger than any of the other terms in the equation. Hence the atmosphere is very close to a state of hydrostatic balance, in which:

$$\frac{\partial p}{\partial z} = -\rho g. \qquad (1.32)$$

This balance only breaks down for small scale phenomena, such as thunderstorm updrafts and flow in the vicinity of very rugged mountain surfaces. On scales greater than around 10 km, hydrostatic balance is usually valid.

The contrast between the vertical scale of the global atmosphere, which can be taken as 7 – 10 km, and its horizontal scale, of around 6000 km, means that the vertical component of velocity is very much smaller than either of the horizontal components. The stable stratification of the atmosphere and the rotation of the system further inhibit vertical motion. This means that several terms involving w in the governing equations, such as the $2\Omega w \cos\phi$ term in the zonal momentum equation, Eq. (1.29a), can be neglected. The result is the so-called 'primitive equation set', which is widely used for numerical weather prediction and global circulation models. The primitive equations on a spherical planet of radius a are set out in Table 1.2 for easy reference. The quantity $f = 2\Omega \sin\phi$ is twice the component of the Earth's angular velocity parallel to the local vertical, known as the 'Coriolis parameter'.

Table 1.2. *The 'primitive' equations*

Equations of motion:

$$\frac{\partial u}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial u}{\partial \lambda} + \frac{v}{a}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} + \frac{uv}{a}\tan\phi = fv - \frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial \lambda} + \mathscr{F}_1, \qquad (1.33a)$$

$$\frac{\partial v}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial v}{\partial \lambda} + \frac{v}{a}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + \frac{u^2}{a}\tan\phi = -fu - \frac{1}{\rho a}\frac{\partial p}{\partial \phi} + \mathscr{F}_2, \qquad (1.33b)$$

Hydrostatic equation:

$$\frac{\partial p}{\partial z} = -\rho g \qquad (1.34)$$

Equation of continuity:

$$\frac{1}{a\cos\phi}\frac{\partial u}{\partial \lambda} + \frac{1}{a\cos\phi}\frac{\partial(v\cos\phi)}{\partial \phi} + \frac{1}{\rho_R}\frac{\partial(\rho_R w)}{\partial z} = 0. \qquad (1.35)$$

Thermodynamic equation:

$$\frac{\partial \theta}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial \theta}{\partial \lambda} + \frac{v}{a}\frac{\partial \theta}{\partial \phi} + w\frac{\partial \theta}{\partial z} = \mathscr{Q}. \qquad (1.36)$$

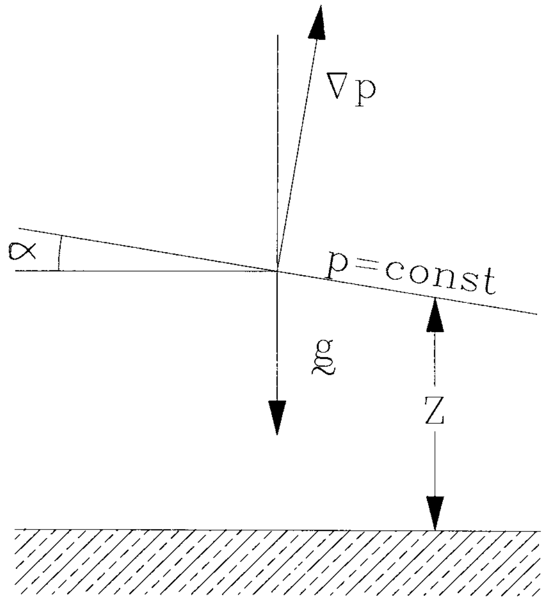

Fig. 1.4. The pressure gradient term in pressure coordinates.

Returning to the hydrostatic equation, Eq. (1.32), the right hand side of the equation is always negative, so that pressure always decreases with height. In fact, integrating from height z to infinity (where $p = 0$):

$$p(z) = \int_z^\infty \rho g \mathrm{d}z. \tag{1.37}$$

That is, the pressure at any level in the atmosphere is simply equal to the weight of the overlying layers of air. The monotonic decrease of pressure with height means that pressure (or indeed, any single valued, monotonic function of pressure) can be used as a vertical coordinate just as well as can height z. The advantage of this procedure is that the equations of motion and the continuity equation are simplified. The disadvantage is that the lower boundary condition is rendered more complicated.

The principal simplification arises in the pressure gradient terms. Consider Fig. 1.4. An isobaric surface will be nearly, though not exactly, horizontal; denote the angle between the normal to the pressure surface and the vertical by α, and denote the magnitude of ∇p by $\partial p/\partial s$; α is typically less than 10^{-3}. From hydrostatic balance,

$$\frac{\partial p}{\partial s} \cos \alpha = \rho g, \tag{1.38}$$

so that the horizontal components of the pressure gradient become

$$-\frac{1}{\rho}\frac{\partial p}{\partial s}\sin\alpha = -g\tan\alpha. \tag{1.39}$$

But $\tan\alpha$ is simply the slope of the isobaric surface $|(\partial Z/\partial x,\ \partial Z/\partial y)|$, where Z denotes the height of the isobaric surface. It follows that in pressure coordinates, the horizontal components of the pressure gradient force can be written:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial Z}{\partial x},\ -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial Z}{\partial y}. \tag{1.40}$$

Using pressure as a vertical coordinate, vertical advection terms such as $w\ \partial Q/\partial z$ transform to

$$w\frac{\partial Q}{\partial z} = \omega\frac{\partial Q}{\partial p}, \tag{1.41}$$

where $\omega = \mathrm{D}p/\mathrm{D}t$ is the pressure coordinate vertical velocity. The pressure vertical velocity is approximately related to the geometric vertical velocity by

$$\omega \approx -\rho g w. \tag{1.42}$$

Similarly, using the hydrostatic relationship, the continuity equation transforms to

$$\nabla\cdot\mathbf{v} + \frac{\partial\omega}{\partial p} = 0. \tag{1.43}$$

Here, the vector $\mathbf{v}$ denotes the *horizontal* component of the velocity vector, $(u, v, 0)$.

The lower boundary condition, which in geometrical coordinates is simply $w = \mathbf{v}\cdot\nabla h$, h being the height of the surface, is considerably less straightforward in pressure coordinates. In the first place, the pressure at the ground fluctuates, so that the boundary moves. In the second, the surface of the Earth is not a coordinate surface. It is sometimes enough to apply a boundary condition $\omega = 0$ at $p = p_R$, but this is certainly not adequate for numerical modelling purposes.

The 'sigma coordinate' system is widely used in numerical models, and combines the simple form of the pressure gradient force in pressure coordinates with the straightforward lower boundary condition of geometric coordinates. Define the vertical coordinate as

$$\sigma = p/p_s, \tag{1.44}$$

where p_s is the actual surface pressure. The Earth's surface is therefore the surface $\sigma = 1$. The vertical advection terms can be written:

$$w\frac{\partial Q}{\partial z} = \dot{\sigma}\frac{\partial Q}{\partial \sigma}, \tag{1.45}$$

where $\dot{\sigma} = \mathrm{D}\sigma/\mathrm{D}t$ is the equivalent of vertical velocity. The boundary conditions are simply $\dot{\sigma} = 0$ at $\sigma = 0$ and $\sigma = 1$. The continuity equation is rendered more complicated; it becomes a prognostic equation for surface pressure:

$$\frac{\partial p_s}{\partial t} + \nabla \cdot (p_s \mathbf{v}) + p_s\frac{\partial \dot{\sigma}}{\partial \sigma} = 0. \tag{1.46}$$

This rather strange equation relates the rate of change of surface pressure to the divergence at an arbitrary level in the atmosphere. The vertical advection term relates the flow at the chosen level to that at other levels. The equation may be integrated with respect to σ; making use of the boundary conditions yields:

$$\frac{\partial p_s}{\partial t} + \int_0^1 \nabla \cdot (p_s \mathbf{v})\mathrm{d}\sigma = 0. \tag{1.47}$$

For analytical work, the extra complications of the sigma coordinate system makes it impractical. It is usually reserved for numerical integration.

It is sometimes helpful, especially for work in the stratosphere, to introduce a 'pseudo-height', proportional to $\ln(p)$:

$$z' = -H\ln(p), \tag{1.48}$$

where H is a constant called the 'pressure scale height'. Integration of the hydrostatic relation shows that this is equal to geometric height for the special case of an atmosphere whose temperature does not change with height. The temperature in the lower stratosphere varies only weakly with height. An equation set based on $\ln(p)$ retains the advantages of a simple pressure gradient term, but expands the rarefied upper levels of the atmosphere.

Other, more complicated, vertical coordinates, such as the potential temperature θ ('isentropic coordinates'), as well as hybrid coordinates which are defined to be sigma coordinates near the ground and pressure coordinates at higher levels, will not be discussed further here. The interested reader is referred to texts on dynamical meteorology for a discussion of generalized vertical coordinates.

1.6 Vorticity

Taking the curl of the momentum equation gives a vorticity equation where relative vorticity $\boldsymbol{\xi} = \nabla \times \mathbf{u}$. In some ways, the vorticity equation is a

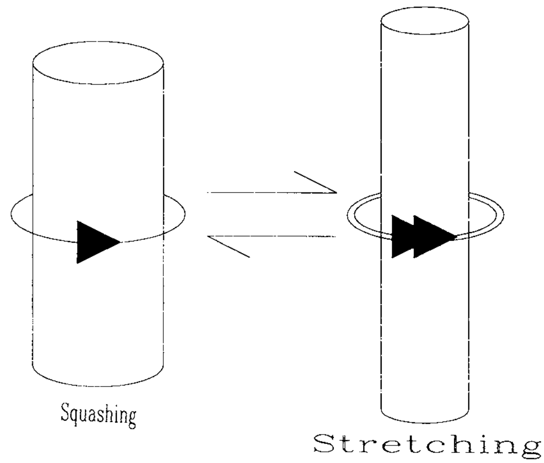

Fig. 1.5. Vortex stretching mechanism for generating relative vorticity.

more convenient way of expressing atmospheric dynamics since, in pressure coordinates, it involves no explicit reference to the pressure field. After some manipulation, the full vorticity equation may be written:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\xi} = (2\boldsymbol{\Omega} + \boldsymbol{\xi}) \cdot \nabla \mathbf{u} - (2\boldsymbol{\Omega} + \boldsymbol{\xi}) \nabla \cdot \mathbf{u} + \frac{\nabla \rho \times \nabla p}{\rho^2} + \nabla \times \mathscr{F}. \quad (1.49)$$

In fact, the large scale dynamics of the atmosphere are determined by the vertical component of the vorticity. The numerically larger horizontal components play a less active role in determining the evolution of the meteorological flow. The vertical component of Eq. (1.49) is most simply written in pressure coordinates. The third term on the right hand side is zero since we are concerned with the component of vorticity perpendicular to pressure surfaces. The second term on the right hand side is zero by virtue of continuity, Eq. (1.35), and so the result is:

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \zeta = (f + \xi) \frac{\partial \omega}{\partial p} + \mathbf{k} \cdot (\nabla \times \mathscr{F}). \quad (1.50)$$

Meteorologists frequently refer to this vertical component of the relative vorticity simply as 'vorticity'. The Coriolis parameter f is sometimes referred to as the 'planetary vorticity'. The absolute vorticity, $(f + \xi)$, has a simple physical interpretation. It is simply twice the angular velocity of the air parcel about a vertical axis. On a rapidly rotating planet such as the Earth, one might suspect that this rotation is generally dominated by the rotation of the planet itself. This turns out to be the case, a fact which will be used in Section 1.7 to derive the 'quasi-geostrophic' approximation.

The crucial term in governing changes to the vorticity is the first remaining term on the right hand side of Eq. (1.50), which represents the generation of vorticity by stretching of vortices as a result of vertical motions. If the vertical velocity stretches a column of air, the column will assume a smaller radius and will rotate more rapidly about its vertical axis, that is, its vorticity will increase. Conversely, squashing of the column will reduce its vorticity. Figure 1.5 illustrates vortex stretching.

Friction generally acts to reduce the relative vorticity towards zero. Newtonian friction can be written in terms of vorticity as

$$\mathbf{k} \cdot (\nabla \times \mathcal{F}) = -\frac{\xi}{\tau_D}, \tag{1.51}$$

a term which represents an exponential decay of the relative vorticity towards zero in the absence of other processes. In fact, Ekman layers, which result from laminar flow over a solid rotating boundary, give rise to precisely such a dissipative term, which arises because the friction near the surface induces small vertical velocities at the top of the boundary layer. The magnitude of these vertical motions is proportional to the relative vorticity just above the boundary layer, and their direction is such as to induce vortex squashing when the interior relative vorticity is positive, and vice versa. Pedlosky gives a good account of this 'spin up' process. The structure of the Ekman layer is a poor approximation to the observed structure of the atmospheric boundary layer, but this effect of spinning up of the interior vorticity is a helpful qualitative model of the effect that boundary layer friction has on vorticity.

1.7 The quasi-geostrophic approximation

Away from the equator, the large scale meteorological flow is close to a state of geostrophic balance. That is, the dominant terms in the horizontal momentum equations, Eqs. (1.33a, b), are the Coriolis terms and the pressure gradient terms. Thus, the 'geostrophic' velocity field is determined by the gradients of the geopotential height:

$$u_g = -\frac{g}{f}\frac{\partial Z}{\partial y}, \quad v_g = \frac{g}{f}\frac{\partial Z}{\partial x}. \tag{1.52}$$

Differentiating these relationships and making use of the hydrostatic equation, Eq. (1.32) and the definition of potential temperature, the vertical variations of u_g and v_g are related to horizontal variations of potential temperature:

$$\frac{\partial u_g}{\partial p} = \frac{h}{f}\frac{\partial \theta}{\partial y}, \quad \frac{\partial v_g}{\partial p} = -\frac{h}{f}\frac{\partial \theta}{\partial x} \tag{1.53}$$

where

$$h(p) = \frac{R}{p}\left(\frac{p}{p_R}\right)^{\kappa}. \tag{1.54}$$

Equation (1.53) shows that the geostrophic wind and temperature field are not independent, but are related in a state of 'thermal wind balance'. A crucial aspect of any process which (for instance) changes the temperature field is that there must be a compensating adjustment of the wind field in order to preserve thermal wind balance. Examples of this adjustment process will be discussed in Chapters 4 and 5. As an alternative to Eq. (1.53), the variation of geostrophic vorticity with height can be written:

$$\frac{\partial \xi_g}{\partial p} = -\frac{h}{f}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right). \tag{1.55}$$

These relationships between the geostrophic velocity (or vorticity) fields and the geopotential height and temperature fields can be used to simplify the governing equations, giving an approximate set which is called the 'quasi-geostrophic' equation set. This has now fallen out of favour as an equation set for modelling the atmospheric circulation since it is not uniformly valid as one approaches the equator; but it remains of great value in diagnosing, and gaining insight, into the dominant dynamical processes in the midlatitude and subtropical regions.

Although Eq. (1.52) represents the dominant terms in the momentum equations, it is of little use for predicting the evolution of the flow. The time derivative terms have been dropped, and so the approximated equations are simply diagnostic, relating the velocity to the pressure field. To determine the evolution of these fields, some ageostrophic effects must be retained. The approach in this section will be via the vorticity equation. First, it is necessary to examine the conditions for which geostrophic balance will hold.

Suppose that a typical horizontal velocity has magnitude U and that it varies over a characteristic length scale L. In the Earth's midlatitudes, U is around 10 m s^{-1} and L might be of the order of 10^6 m. Then the typical magnitude of the horizontal advection terms in the momentum equations will be U^2/L. The magnitude of the Coriolis term will be fU. The ratio of the two is called the 'Rossby number' Ro:

$$\frac{U^2/L}{fU} = \frac{U}{fL} = Ro. \tag{1.56}$$

A necessary condition for geostrophic balance to be achieved is that the Rossby number be small. Other conditions are that the friction term be small, and that the trajectories of fluid elements be only gently curved. For